

# Schwinger mechanism enhanced by the Nielsen–Olesen instability

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## Abstract

We discuss gluon production by the Schwinger mechanism in collinear color-electric and magnetic fields which may be realized in pre-equilibrium stages of ultra-relativistic heavy-ion collisions. Fluctuations of non-Abelian gauge fields around a purely color-magnetic field contain exponentially growing unstable modes in a longitudinally soft momentum region, which is known as the Nielsen–Olesen instability. With a color-electric field imposed parallelly to the color-magnetic field, we can formulate this instability as the Schwinger mechanism. This is because soft unstable modes are accelerated by the electric fields to escape from the instability condition. Effects of instability remain in the transverse spectrum of particle modes, leading to an anomalously intense Schwinger particle production.

## 1 Introduction

Multi-particle production in strong fields is a typical unstable phenomenon that can be seen in extreme situations such as ultra-relativistic heavy-ion collisions, high-energy astrophysical objects, and possibly high-intensity laser. This phenomenon is in general nonperturbative even for weakly interacting systems because the coupling with external strong fields compensates the weakness of the interaction. A straightforward description of such an instability may be given by the effective action of the strong fields. A famous example is the Euler-Heisenberg action, which is an effective action of strong electromagnetic fields due to electron's one-loop contribution. However, such an effective action does not directly tell us the information of dynamical processes which cause instability. To obtain dynamical pictures of instabilities, we have to resort to other methods which allow us to describe the degrees of freedom relevant for instabilities.

When the produced particles are normalizable and asymptotically stable modes, one can describe particle production within the canonical quantization. This is indeed the case with the Schwinger mechanism [1] where a pair of a charged particle and an antiparticle is created in a strong field.

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On the other hand, when a system contains genuine unstable modes which are unnormalizable and keep growing even at asymptotic regions, one cannot apply the canonical quantization to those modes. When the mode amplitude grows in an unnormalizable way, this mode should be treated as not a particle but a field. However, if the instability lasts only for a finite duration, we have, after the instability, an asymptotic region where stable particle modes can be defined. Then we can provide particle interpretation of the modes by the canonical quantization, and therefore the instability can be viewed as a particle production phenomena. In fact, this situation is seen in the parametric resonance, where the time-dependent external field induces instability only for a finite time interval, and particle number is defined in the free regions [2, 3].

Notice that one encounters both of the cases in non-Abelian Yang-Mills theories; The one-loop effective action of non-Abelian gauge fields under constant electric or magnetic background has an imaginary part, which indicates the existence of instabilities [4]. The instability in a purely electric field consists of normalizable and asymptotically stable modes, and thereby can be interpreted as particle productions [5, 6]. One can formulate it as a non-Abelian analog of the Schwinger mechanism [7]. In contrast, the instability in a purely magnetic background is caused by rapid growth of particular fluctuations and thus cannot be formulated as particle production unless the magnetic field is turned off in finite time so that the modes are stabilized. This “Nielsen–Olesen (N-O) instability” [8] is characteristic of non-Abelian gauge fields because the properties of self-interaction and spin-1 are necessary.

The instabilities in these two limiting cases have been individually studied in the context of early time dynamics of ultra-relativistic heavy-ion collisions [9, 10, 11, 12] (see also Ref. [13]). However, the strong non-Abelian fields that appear in reality (called a ‘glasma’) would consist of longitudinally extended flux tubes in which both the electric and magnetic fields are non-vanishing and collinear along the beam axis [14]. It is important to investigate particle production in this configuration to understand time evolution of created matter towards a thermalized quark-gluon plasma. In this Letter, we study, within the canonical quantization approach, the gauge field instability (gluon production) when both electric and magnetic background fields are present.

## 2 Formalism

We consider the  $SU(N)$  pure Yang-Mills theory, and decompose the gauge field  $A_\mu^a$  ( $a = 1, \dots, N^2 - 1$ ) into a classical background  $\bar{A}_\mu^a$  and a quantum fluctuation  $\mathcal{A}_\mu^a$  around the background:  $A_\mu^a = \bar{A}_\mu^a + \mathcal{A}_\mu^a$ . We take a covariantly-constant background field [15] satisfying

$$\bar{D}_\mu \bar{F}^{a\mu\nu} = 0, \quad (1)$$

where  $\bar{D}_\mu$  is a covariant derivative with respect to the background:  $\bar{D}_\mu \chi^a = (\partial_\mu \delta^{ac} - g f^{abc} \bar{A}_\mu^b) \chi^c$ , and  $\bar{F}_{\mu\nu}^a$  is a classical field strength:  $\bar{F}_{\mu\nu}^a = \partial_\mu \bar{A}_\nu^a - \partial_\nu \bar{A}_\mu^a - g f^{abc} \bar{A}_\mu^b \bar{A}_\nu^c$ . We employ the background covariant gauge [7]:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\zeta} (\bar{D}_\mu \mathcal{A}^{a\mu})^2 - i (\bar{D}^\mu \bar{C}^a) D_\mu C^a, \quad (2)$$

where  $\zeta$  is a gauge parameter (to be set as  $\zeta = 1$ ) and  $C^a$  ( $\bar{C}^a$ ) is the Faddeev–Popov (anti-)ghost. The last covariant derivative  $D_\mu$  contains both the background  $\bar{A}_\mu^a$  and the fluctuation  $\mathcal{A}_\mu^a$ . In order to be consistent with the one-loop calculation of the effective action, we retain in the Lagrangian up to quadratic terms with respect to the quantum fields. Then, the equations of motion for the quantum fields are linearized as follows:

$$\bar{D}_\nu \bar{D}^\nu \mathcal{A}^{a\mu} - 2gf^{abc} \bar{F}^{b\mu\nu} \mathcal{A}_\nu^c = 0, \quad (3)$$

$$\bar{D}_\mu \bar{D}^\mu C^a = 0, \quad \bar{D}_\mu \bar{D}^\mu \bar{C}^a = 0. \quad (4)$$

We consider the classical background field which gives spatially constant field strength. Then, one can express the field strength as [15, 9]

$$\bar{F}_{\mu\nu}^a = \bar{F}_{\mu\nu} n^a, \quad (5)$$

where  $\bar{F}_{\mu\nu}$  is an Abelian field strength and  $n^a$  is a color vector, normalized as  $n^a n^a = 1$ . The field strength (5) is given by the gauge potential

$$\bar{A}_\mu^a = \bar{A}_\mu n^a, \quad (6)$$

where  $\bar{A}_\mu$  is an Abelian gauge field giving  $\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu$ . This is actually a solution to Eq. (1). The global residual gauge symmetry allows us to rotate the color vector  $n^a$  into the Cartan subspace of  $SU(N)$ :  $Un^a T^a U^\dagger = \tilde{n}^A H^A$ , where  $U$  is a constant  $SU(N)$  matrix,  $T^a$  are generators of  $SU(N)$  and  $H^A$  ( $A = 1, 2, \dots, N-1$ ) are elements of the Cartan subspace:  $H^A \in \{T^a | [T^a, T^b] = 0\}$ .

We expand the  $SU(N)$  space by  $H^A$  and  $v_\alpha^a T^a$ , instead of  $T^a$ . Here,  $v_\alpha^a$  are eigenvectors of  $\text{ad}\{H^A\}^{bc}$  (i.e.,  $H^A$  in the adjoint representation):

$$\text{ad}\{H^A\}^{bc} v_\alpha^c = \alpha^A v_\alpha^b. \quad (7)$$

The corresponding eigenvalues  $\alpha^A$  may be regarded as an  $(N-1)$ -dimensional vector, which is called the root vector. According to this Cartan decomposition, we re-organize the quantum fields  $\mathcal{A}_\mu^a$ ,  $C^a$  and  $\bar{C}^a$  as

$$\{\mathcal{A}_\mu^a\} \Rightarrow \{a_\mu^A\} \oplus \{W_\mu^\alpha \equiv v_\alpha^a \mathcal{A}_\mu^a\}, \quad (8)$$

$$\{C^a\} \Rightarrow \{c^A\} \oplus \{\eta^\alpha \equiv v_\alpha^a C^a\}, \quad (9)$$

$$\{\bar{C}^a\} \Rightarrow \{\bar{c}^A\} \oplus \{\bar{\eta}^\alpha \equiv v_\alpha^a \bar{C}^a\}. \quad (10)$$

This decomposition simplifies the interaction between the fluctuations  $\mathcal{A}_\mu^a$ ,  $C^a$ ,  $\bar{C}^a$  and the classical field  $\bar{A}_\mu^a$ . The fields in the Cartan subspace,  $\phi^A \equiv a_\mu^A, c^A, \bar{c}^A$ , are free from the background field:  $\bar{D}_\mu \phi^A = \partial_\mu \phi^A$ , and thus we need not consider these fields in the following. For the off-diagonal fields,  $\Phi^\alpha \equiv W_\mu^\alpha, \eta^\alpha, \bar{\eta}^\alpha$ , the covariant derivative reduces to an Abelian form as

$$\bar{D}_\mu \Phi^\alpha = (\partial_\mu + ie_\alpha \bar{A}_\mu) \Phi^\alpha, \quad (11)$$

with  $e_\alpha = -\tilde{n}^A \alpha^A g$ . Gauge invariance of  $e_\alpha$  has been explicitly verified for  $SU(3)$  [16]. One can discuss quark production in a similar way [17].

In terms of these new fields, the equations of motion (3), (4) are rewritten as

$$(\partial_\nu + ie_\alpha \bar{A}_\nu)^2 W_\mu^\alpha + 2ie_\alpha \bar{F}_{\mu\nu} W^{\alpha\nu} = 0, \quad (12)$$

$$(\partial_\mu + ie_\alpha \bar{A}_\mu)^2 \eta^\alpha = 0, \quad (\partial_\mu + ie_\alpha \bar{A}_\mu)^2 \bar{\eta}^\alpha = 0. \quad (13)$$

Solutions of these equations may be expanded by normal modes:

$$W_\mu^\alpha(x) = \sum_{\sigma=\pm, L, S} \sum_n [f_n^{(\sigma)}(x) a_n^{(\sigma)\alpha} + f_n^{(\sigma)*}(x) b_n^{(\sigma)\alpha\dagger}] \epsilon_\mu^{(\sigma)},$$

$$\eta^\alpha(x) = \sum_n [h_n(x) c_n^\alpha + h_n^*(x) d_n^{\alpha\dagger}], \quad (14)$$

$$\bar{\eta}^\alpha(x) = \sum_n [h_n(x) \bar{c}_n^\alpha + h_n^*(x) \bar{d}_n^{\alpha\dagger}], \quad (15)$$

where  $n$  is quantum number(s) characterizing the mode, such as momentum  $\mathbf{p}$ , and  $\sigma(=\pm, L, S)$  denotes polarization.  $f_n^{(\sigma)}(x) \epsilon_\mu^{(\sigma)}$  and  $h_n(x)$  are c-number solutions of the equations of motion (12) and (13), respectively.  $\epsilon_\mu^{(\sigma)}$  are polarization vectors satisfying the orthonormal condition  $\epsilon_\mu^{(\sigma)*} \epsilon^{(\tau)\mu} = -\tilde{\eta}^{\sigma\tau}$  and the completeness condition  $\sum_{\sigma, \tau} \epsilon_\mu^{(\sigma)} \tilde{\eta}^{\sigma\tau} \epsilon_\nu^{(\tau)*} = -g_{\mu\nu}$  with the metric  $\tilde{\eta}^{\sigma\tau}$  such that  $\tilde{\eta}^{++} = \tilde{\eta}^{--} = \tilde{\eta}^{LS} = \tilde{\eta}^{SL} = 1$  and other components are vanishing. We identify the solutions  $f_n^{(\sigma)}(x) \epsilon_\mu^{(\sigma)}$  and  $h_n(x)$  as positive frequency modes, and  $f_n^{(\sigma)*}(x) \epsilon_\mu^{(\sigma)}$  and  $h_n^*(x)$  as negative frequency modes. The positive frequency solutions are to satisfy the following normalization conditions:

$$-g^{\mu\nu} (f_n^{(\sigma)} \epsilon_\mu^{(\sigma)}, f_m^{(\tau)} \epsilon_\nu^{(\tau)}) = \tilde{\eta}^{\sigma\tau} \delta_{nm}, \quad (16)$$

$$(h_n, h_m) = \delta_{nm}, \quad (17)$$

where the inner product is defined as

$$(\phi_1, \phi_2) = i \int d^3x (\phi_1^* \cdot \bar{D}_0 \phi_2 - \bar{D}_0 \phi_1^* \cdot \phi_2). \quad (18)$$

Notice that the norms of the negative-frequency modes have the sign opposite to those of the positive-frequency modes, and the positive and negative frequency modes are orthogonal to each other. With these orthonormal conditions, the canonical commutation relations imposed on new fields  $W_\mu^\alpha, \eta^\alpha, \bar{\eta}^\alpha$  yield the following commutation relations:

$$[a_n^{(\sigma)\alpha}, a_m^{(\tau)\beta\dagger}] = [b_n^{(\sigma)\alpha}, b_m^{(\tau)\beta\dagger}] = \tilde{\eta}^{\sigma\tau} \delta_{\alpha\beta} \delta_{nm}, \quad (19)$$

$$\{c_n^\alpha, \bar{c}_m^{\beta\dagger}\} = \{d_n^\alpha, \bar{d}_m^{\beta\dagger}\} = i \delta_{\alpha\beta} \delta_{nm}. \quad (20)$$

Now we can regard the operators  $a_n^{(\sigma)\alpha}, b_n^{(\sigma)\alpha}, c_n^\alpha, d_n^\alpha, \bar{c}_n^\alpha$  and  $\bar{d}_n^\alpha$  as annihilation operators for particles or antiparticles.

### 3 Instability and gluon production

With the canonical framework defined above, we investigate the quantum dynamics of fluctuations in the collinear color-electric and magnetic background field:

$$\mathbf{E}^a = (0, 0, E) n^a, \quad \mathbf{B}^a = (0, 0, B) n^a, \quad (21)$$

which is realized by  $\bar{A}^\mu = (0, -By, 0, -Et)$ . It is convenient to choose the polarization vectors as eigenvectors of the field strength  $\bar{F}^{\mu\nu}$ . Then, all the equations that the mode functions  $f_n^{(\sigma)}(x)$  and  $h_n(x)$  follow can be put together into a Klein–Gordon-like equation

$$\left[ (\partial_\nu + ie_\alpha \bar{A}_\nu)^2 + m^2 \right] \Phi(t, \mathbf{x}; m^2) = 0, \quad (22)$$

where  $\Phi = f_n^{(\sigma)}, h_n$ . In the following, the subscript  $\alpha$  of  $e_\alpha$  will be often omitted. The “mass squared”  $m^2$  represents the effects of spin-electromagnetic field interaction and is given by

$$m^2 = \begin{cases} \mp 2eB & \text{for gluon modes with } \sigma = \pm \\ 2ieE & \text{for gluon modes with } \sigma = L \\ -2ieE & \text{for gluon modes with } \sigma = S \\ 0 & \text{for ghost modes.} \end{cases} \quad (23)$$

Because the  $\sigma = L, S$  modes acquire pure imaginary  $m^2$ , we should suppose that the negative frequency solution for  $\sigma = L$  mode is  $f_n^{(S)*}$ , and *vice versa*.

Let us first consider the case where we have only a color-electric field ( $E \neq 0, B = 0$ ). Ambjørn and Hughes [7] have studied the Schwinger particle production in the canonical quantization approach in the background covariant gauge (2). One can find exact solutions of Eq. (22) and show that all the mode functions satisfy the normalization conditions (16) and (17). Therefore, in the pure electric field the canonical quantization can be completed to give particle interpretation of the fields. However, under electric fields, selection of the positive and negative frequencies is not unique. There are infinite numbers of solutions satisfying the conditions (16) and (17). If there is no electric field, we can choose the positive and negative frequencies uniquely as  $e^{\mp i\omega_p t}$  respecting the translational invariance in time. In contrast, under electric fields, the system loses that invariance, so that we cannot select a specific set of solutions without other criterion. To decide proper solutions, asymptotic WKB criterion is employed [5, 7, 6]. In this criterion, the positive and negative frequency solutions are selected so that they behave at asymptotic region,  $|t| \rightarrow \infty$ , as  $\Phi(t, \mathbf{x}) \sim \exp(iS_{cl})$  with  $S_{cl}$  being the classical action of a charged particle under the electric field. As a direct consequence of particle production, positive and negative frequency solutions defined at  $t \rightarrow -\infty$  and those at  $t \rightarrow +\infty$  are different. The former, which are referred to as in-solutions, are

$$\Phi_{\mathbf{p}}^{\text{in}}(t, \mathbf{x}; m^2) = \frac{e^{-\frac{\pi}{4}a_\perp}}{(2eE)^{\frac{1}{4}}} D_{ia_\perp - \frac{1}{2}} \left( -e^{-\frac{\pi}{4}i\xi} \right) \frac{e^{i\mathbf{p} \cdot \mathbf{x}}}{\sqrt{(2\pi)^3}}, \quad (24)$$

with  $a_\perp = \frac{m_\perp^2}{2eE}$ ,  $m_\perp^2 = m^2 + p_x^2 + p_y^2$  and  $\xi = \sqrt{\frac{2}{eE}}(p_z + eEt)$ .  $D_\nu(x)$  is a parabolic cylinder function. The latter, out-solutions, are

$$\Phi_{\mathbf{p}}^{\text{out}}(t, \mathbf{x}; m^2) = \frac{e^{-\frac{\pi}{4}a_\perp}}{(2eE)^{\frac{1}{4}}} D_{-ia_\perp - \frac{1}{2}} \left( e^{\frac{\pi}{4}i\xi} \right) \frac{e^{i\mathbf{p} \cdot \mathbf{x}}}{\sqrt{(2\pi)^3}}. \quad (25)$$

Both solutions have the structure of plane waves distorted by the electric field. Associated with these two kinds of mode functions, we get two kinds of particle creation and annihilation operators,  $a_{\mathbf{p}}^{(\sigma)\text{in}}$  and  $a_{\mathbf{p}}^{(\sigma)\text{out}}$ , etc. Accordingly, two kinds of vacua  $|0, \text{in}\rangle$  and  $|0, \text{out}\rangle$  are defined. The creation and annihilation operators of in and out are related

with each other via Bogoliubov transformations. Then one can relate the in-vacuum and the out-vacuum and calculate the vacuum persistence probability  $|\langle 0, \text{out} | 0, \text{in} \rangle|^2$ . It has been shown that contributions from the unphysical modes,  $\sigma = \text{L, S}$ , and those from the ghost modes are totally canceled out, and the pair creation rate  $w$  defined by  $|\langle 0, \text{out} | 0, \text{in} \rangle|^2 = \exp(-VTw)$  with  $V$  and  $T$  being space and time volume, respectively, is just twice as large as that of a massless charged scalar field [7]. Also, we can calculate the momentum distribution of out-particles condensed in the in-vacuum, in other words, the momentum distribution of particles which are created during infinite-time imposition of the electric field. For the physical modes,  $\sigma = \pm$ , we find

$$\langle 0, \text{in} | a_{\mathbf{p}}^{(\sigma)\alpha \text{out}\dagger} a_{\mathbf{p}}^{(\sigma)\alpha \text{out}} | 0, \text{in} \rangle = \exp\left(-\frac{\pi p_{\perp}^2}{e_{\alpha} E}\right) \frac{V}{(2\pi)^3}, \quad (26)$$

which depends only on the transverse momentum  $p_{\perp}^2 = p_x^2 + p_y^2$ .<sup>\*1</sup> This result is the same as that of a massless charged scalar field. In contrast, for the unphysical modes,  $\sigma = \text{L, S}$ , and the ghost modes, the expectation is vanishing because of their unusual commutation relations (19) and (20).

In contradistinction to the electric field, the magnetic field has no dynamical effect. It just discretizes the transverse momenta into the Landau levels as

$$p_{\perp}^2 \rightarrow (2n+1)eB \quad (n = 0, 1, 2, \dots). \quad (27)$$

Under the pure magnetic field ( $E = 0, B \neq 0$ ), one can find a solution of Eq. (22) as

$$\Phi_{\mathbf{p}}(t, \mathbf{x}; m^2) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i\omega_{\mathbf{p}} t} \varphi_n(y) \frac{1}{2\pi} e^{ip_x x + ip_z z}, \quad (28)$$

where  $\omega_{\mathbf{p}} = \sqrt{m^2 + p_z^2 + (2n+1)eB}$  and  $\varphi_n(y) \propto D_n\left(\sqrt{\frac{2}{eB}}(eBy + p_x)\right)$ . The spin-magnetic field interaction (23) (with  $E = 0$ ) shifts the Landau levels as shown in Fig. 1. Notice that the transverse mass squared defined as  $m_{\perp}^2 = m^2 + (2n+1)eB$  is negative for the lowest Landau level  $n = 0$  with  $\sigma = +$  mode, and the one-particle energy reads  $\omega_{p_z, n=0} = \sqrt{p_z^2 - eB}$ . This is pure imaginary when  $|p_z| < \sqrt{eB}$ . Therefore, the mode functions show exponential growth  $\sim \exp\left(\sqrt{eB - p_z^2} t\right)$ . This is the N-O instability [8]. When this instability occurs, the normalization (16) cannot hold, so that we can no longer gain particle interpretation of the field. If the magnetic field is not constant in time and if there are asymptotically free regions where particle modes can be defined by the canonical quantization, the N-O instability could be described as particle production phenomena. However, as far as the authors know, there has been no explicit demonstration of it. We will show that we can define asymptotically stable particle modes even in the presence of the constant magnetic field provided there is a constant electric field parallel to the magnetic field.

Notice that the N-O instability occurs only at low longitudinal momentum region  $|p_z| < \sqrt{eB}$ . Under the collinear electric and magnetic fields ( $E \neq 0, B \neq 0$ ), particles are accelerated by the electric field and their momenta increase in time as  $p_z = eEt + \text{const.}$

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<sup>\*1</sup>As discussed in Ref. [6],  $p_z$ -dependence of the distribution cannot be obtained by the method of the asymptotic WKB criterion.

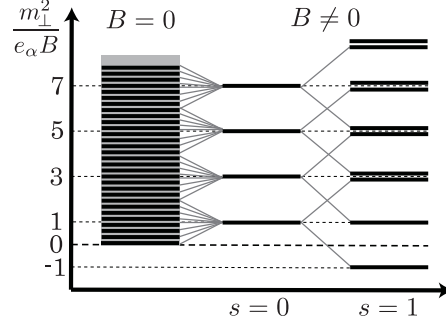


Figure 1: A schematic of the Landau level and the level splitting due to spin-magnetic field interaction

Therefore, even if a particle lies in the unstable momentum region  $|p_z| < \sqrt{eB}$  at some time, it escapes from there unfailingly at later time. Owing to this mechanism, the exponential growth of the mode function is regularized in the presence of the color-electric field, and we can carry out the canonical quantization and obtain the particle interpretation for all the modes. Indeed, exact solutions of Eq. (22) under the collinear fields

$$\Phi_{\mathbf{p}}^{\text{in}}(t, \mathbf{x}; m^2) = \frac{e^{-\frac{\pi}{4}a_n}}{(2eE)^{\frac{1}{4}}} D_{ia_n - \frac{1}{2}}(-e^{-\frac{\pi}{4}i\xi}) \varphi_n(y) \frac{e^{ip_x x + ip_z z}}{2\pi}, \quad (29)$$

$$\Phi_{\mathbf{p}}^{\text{out}}(t, \mathbf{x}; m^2) = \frac{e^{-\frac{\pi}{4}a_n}}{(2eE)^{\frac{1}{4}}} D_{-ia_n - \frac{1}{2}}(e^{\frac{\pi}{4}i\xi}) \varphi_n(y) \frac{e^{ip_x x + ip_z z}}{2\pi}, \quad (30)$$

with  $a_n = (n + \frac{1}{2}) \frac{B}{E}$  do not diverge at  $t \rightarrow +\infty$  and satisfy the normalization conditions (16) and (17). This is true even for the lowest Landau mode with  $n = 0$  and  $\sigma = +$ . Since the magnetic field has no dynamic effect, the selection of the in- and out- mode functions is not affected by it. Therefore, the Bogoliubov relations are almost the same as those in the pure electric field. The differences are the emergence of the Landau levels (27) and the level splitting between the two transverse modes (see Eq. (23)). Because  $m^2$  for the unphysical modes are independent of  $B$ , cancellation between those modes again holds even in the presence of the magnetic field. Actually, the pair creation rate has contributions only from the two physical transverse modes:

$$w = \frac{N}{2} \frac{g^2 EB}{(2\pi)^2} \sum_{\sigma=\pm} \sum_{n=0}^{\infty} \ln \left[ 1 + e^{-\pi(2n+1-2\sigma)\frac{B}{E}} \right]. \quad (31)$$

Furthermore, the in-vacuum expectations of out-number operators for the unphysical modes are all vanishing also in the presence of the magnetic field. Those for the physical transverse modes are

$$\langle 0, \text{in} | a_{p_z, n}^{(\pm)\alpha \text{out}\dagger} a_{p_z, n}^{(\pm)\alpha \text{out}} | 0, \text{in} \rangle = e^{-\pi(2n+1\mp 2)\frac{B}{E}} \frac{V}{(2\pi)^3}. \quad (32)$$

In Eqs. (31) and (32), the lowest level ( $\sigma = +$  and  $n = 0$ ) is worthy of great remark. In this mode, the effective mass square is negative,  $m_{\perp}^2 = -eB$ , so that the index of the exponential factor is positive:

$$\langle 0, \text{in} | a_{p_z, n=0}^{(+)\alpha \text{out}\dagger} a_{p_z, n=0}^{(+)\alpha \text{out}} | 0, \text{in} \rangle = e^{+\pi\frac{B}{E}} \frac{V}{(2\pi)^3}. \quad (33)$$

This means that an anomalously large number of gluons are produced in the lowest mode when compared with the usual Schwinger mechanism of QED or scalar QED. This is because the field fluctuations amplified by the N-O instability are converted to real particles by the electric field.

The collinear color electromagnetic field (21) may be realized in flux tubes created at the initial stage of heavy-ion collisions [14]. The strongly enhanced particle production causes rapid decay of those coherent fields into particle degrees of freedom, which would be followed by the formation of a quark-gluon plasma.

## 4 Discussion

Our analysis has relied on the linear approximation. If the field amplification by the N-O instability continues for a long time, the linear approximation would get invalid. A typical time scale of the N-O instability is given by the inverse of the growth rate  $t_{\text{NO}} \sim 1/\sqrt{eB}$ . Therefore, the time scale when the non-linear corrections become important would be  $t_{\text{nl}} \gtrsim 1/\sqrt{eB}$ . Actually, it has been demonstrated by numerical calculations that the exponential growth of the gauge fluctuations stops at some time (later than  $t_{\text{NO}}$ ) due to non-linear corrections [18]. Meanwhile, the time scale of the stabilization process by the electric field  $t_{\text{ele}}$  is given from the condition that a particle is accelerated away from the instability condition  $|p_z| < \sqrt{eB}$ , so that  $t_{\text{ele}} \sim \sqrt{eB}/eE$ . If the electric and magnetic fields are the same order of magnitude, which is a situation expected in a glasma,  $t_{\text{ele}}$  is the same order as  $t_{\text{NO}}$ , so that the electric field plays a dominant role to stabilize the N-O instability.

To understand more clearly our result that the N-O instability is stabilized by the electric field and induces the strong particle production, the following simple toy model which describes particle production by an instability may be useful. Let us suppose that, without specification of background fields, a real scalar field suffers an instability during a finite period,  $0 < t < T$ . The mode functions of the field have the following properties: (i) for  $t < 0$  the modes are stable and have the plane wave solutions;  $\frac{1}{\sqrt{2\omega}}e^{\mp i\omega t}$  ( $\omega \geq 0$ ), (ii) for  $0 < t < T$  the modes are unstable and show exponential growth or decrease;  $e^{\pm\gamma t}$  ( $\gamma > 0$ ), and (iii) for  $T < t$  the modes are again stable;  $\frac{1}{\sqrt{2\Omega}}e^{\mp i\Omega t}$  ( $\Omega \geq 0$ ). The canonical quantization can be performed at  $t < 0$  and at  $t > T$ , and correspondingly two kinds of particle definitions of ‘in’ and ‘out’ are obtained. One can find the relation between these two kinds of particle definitions by smoothly connecting the mode solutions in the three time regimes. If the solution  $\frac{1}{\sqrt{2\omega}}e^{-i\omega t}$  for  $t < 0$  is smoothly connected to the linear combination of the solutions for  $0 < t < T$ , and is subsequently connected to those for  $t > T$  as  $\frac{1}{\sqrt{2\Omega}}(\alpha e^{-i\Omega t} + \beta^* e^{i\Omega t})$ , then the particle annihilation (creation) operators of in-particle,  $a^{\text{in}(\dagger)}$ , is related with those of out-particle,  $a^{\text{out}(\dagger)}$ , by the Bogoliubov transformation:  $a^{\text{out}} = \alpha a^{\text{in}} + \beta a^{\text{in}\dagger}$ . One can explicitly confirm the condition  $|\alpha|^2 - |\beta|^2 = 1$ , which guarantees the unitarity, holds irrespective of the values of  $\omega$ ,  $\Omega$ ,  $\gamma$  and  $T$ . The number of particles created by the instability is

$$\langle 0, \text{in} | a^{\text{out}\dagger} a^{\text{out}} | 0, \text{in} \rangle = |\beta|^2 \underset{T \gg 1/\gamma}{\simeq} \frac{\Omega}{16\omega} \left(1 + \frac{\omega^2}{\gamma^2}\right) \left(1 + \frac{\gamma^2}{\Omega^2}\right) e^{2\gamma T}. \quad (34)$$

Thus, the number of created particles increases exponentially with respect to the time



interval of the unstable region,  $T$ . In this simple demonstration, the existence of the stable modes in asymptotic regions has been crucial to formulate the particle production in a canonical way. If we compare this simple model with our original problem of the gluon production, the instability parameter  $\gamma$  corresponds to  $1/t_{\text{NO}} \sim \sqrt{eB}$ . Though there is no explicitly stable region in temporary direction under the constant magnetic field, the electric field provides us with the stable regions in the momentum space, *i.e.*  $|p_z| > \sqrt{eB}$ . Hence,  $T$  should be replaced by  $t_{\text{ele}} \sim \sqrt{eB}/eE$ . By substituting these parameters, we find the exponential factor  $e^{2\gamma T}$  in Eq. (34) roughly reproduces the exact result (33).

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